

# Time-Dependent Multi-Centre Solutions from New Metrics with Holonomy $\text{Sim}(n - 2)$

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## Abstract

The classifications of holonomy groups in Lorentzian and in Euclidean signature are quite different. A group of interest in Lorentzian signature in  $n$  dimensions is the maximal proper subgroup of the Lorentz group,  $\text{Sim}(n - 2)$ . Ricci-flat metrics with  $\text{Sim}(2)$  holonomy were constructed by Kerr and Goldberg, and a single four-dimensional example with a non-zero cosmological constant was exhibited by Ghanam and Thompson. Here we reduce the problem of finding the general  $n$ -dimensional Einstein metric of  $\text{Sim}(n - 2)$  holonomy, with and without a cosmological constant, to solving a set linear generalised Laplace and Poisson equations on an  $(n - 2)$ -dimensional Einstein base manifold. Explicit examples may be constructed in terms of generalised harmonic functions. A dimensional reduction of these multi-centre solutions gives new time-dependent Kaluza-Klein black holes and monopoles, including time-dependent black holes in a cosmological background whose spatial sections have non-vanishing curvature.

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# 1 Introduction

Apart from their obvious intrinsic interest, solutions of Einstein's equations, vacuum or with supergravity sources, representing arbitrary many black holes, naked singularities or p-branes moving in a time-independent or time-dependent background have found many applications in string and M-theory. Such metrics are often called multi-centre metrics because they depend on one or more arbitrary functions which are harmonic with respect to some transverse spatial metric  $g_{ij}$ . In this sense the Einstein equations linearise and one may therefore take a superposition of solutions with delta-function sources.

A detailed examination of these ‘anti-gravitating’ [1] solutions near the sources often reveals that they can be regarded as extreme black holes [2], or products of anti-de-Sitter spacetime with an Einstein space [3], or as singular extreme limits of black holes [3]. In other cases the delta-functions represent magnetic monopoles [4–7] and may be resolved by passing to higher dimensions [8]. In some cases the presence of more than one centre gives rise to new singularities [8–10]. The situation most studied is when the transverse metric  $g_{ij}$  is time independent and flat. The harmonic functions may then be taken to be a superposition of solutions of Green functions for the time-independent Laplace equation on Euclidean space. In many cases these metrics may be generalised to the case where the transverse space is curved but still Ricci flat. However, so far we know of few solutions of this type for which  $g_{ij}$  is an Einstein space

$$R_{ij} = \Lambda g_{ij} , \quad (1.1)$$

The main example is the Anti-De-Sitter generalisation of the Brinkmann solutions given in [11]. These are of the form <sup>1</sup>

$$ds^2 = \frac{n-1}{(-\Lambda y^2)} \left\{ 2dudv + H(u, y, x^a) du^2 + dy^2 + h_{ab} dx^a dx^b \right\} \quad (1.2)$$

with  $a = 1, 2, \dots, (n-3)$ , the metric  $h_{ab}(x)$  being Ricci flat and hence the transverse metric

$$g_{ij} dx^i dx^j = \frac{n-1}{(-\Lambda y^2)} \left\{ dy^2 + h_{ab} dx^a dx^b \right\} \quad (1.3)$$

being an Einstein metric with negative scalar curvature.

The function  $H$  satisfies the Laplace type equation

$$y^{n-2} \partial_y \left( \frac{1}{y^{n-2}} \partial_y H \right) + \nabla_h^2 H = 0 , \quad (1.4)$$

$\nabla_h^2$  being the Laplacian with respect to the metric  $h_{ab}$ .

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<sup>1</sup>The full n-dimensional metric satisfies  $R_{\mu\nu} = \Lambda g_{\mu\nu}$ .

Another example is the multi-domain wall solution [12] which, from the point of view of this paper, is rather degenerate case, since it may more simply be viewed as patches of *AdS* glued together across hypersurfaces.

As well as time-independent solutions, a number of time-dependent solutions are known (see [13–16]), but as far as we know so far these are all in the case where the transverse metric is time independent and Ricci flat.

Time-independent multi-centre metrics may (or may not) be supersymmetric or BPS. That is, considered as solutions of a supergravity theory they may admit Killing spinors [17–20]. Thus, in the vacuum case, they may admit a covariantly-constant commuting spinor field  $\epsilon$ . Time-dependent solutions cannot be supersymmetric, since any Killing spinor field  $\epsilon$  gives rise to a non-spacelike Killing vector field  $\bar{\epsilon}\Gamma^\mu\epsilon$ . However, it can happen [15,16,21] that the time-dependent solution arises from the dimensional reduction of a time-independent solution in one higher dimension that does admit a Killing spinor, and hence a non-spacelike Killing vector field  $\bar{\epsilon}\Gamma^\mu\epsilon$  and that also admits an additional, spacelike, boost Killing vector field. Dimensional reduction with respect to this boost Killing field then gives rise to a time-dependent solution in the lower dimension. Because the Killing spinor  $\epsilon$  is not boost invariant, it does not descend to the lower-dimensional spacetime, which is therefore, unlike its higher-dimensional progenitor, not supersymmetric.

Multi-centre metrics can have, or arise by dimensional reduction from, a metric  $g_{\mu\nu}$ ,  $\mu, \nu = 1, 2, \dots, n$  with a reduced holonomy group. Thus for example in the case that the Killing spinor is covariantly constant  $\nabla_\mu\epsilon = 0$ , the associated Killing vector  $n^\mu = \bar{\epsilon}\Gamma^\mu\epsilon$  is also covariantly constant,  $\nabla_\mu n^\nu = 0$ . A particularly interesting example is when the Killing vector field is null  $n^\mu n_\mu = 0$ . The subgroup of the Lorentz group  $SO(n-1, 1)$  leaving the null vector  $n^\mu$  invariant is the Euclidean group  $E(n-2)$ . Thus a metric admitting a covariantly constant Killing vector has holonomy group  $E(n-1)$  or a subgroup thereof. Such metrics are known as Brinkmann waves [22–24]. A special case are the so-called pp-waves of the form

$$ds^2 = 2dudv + dx^i dx^i + H(u, x^i) du^2, \quad (1.5)$$

where the Ricci flat condition becomes

$$\partial_i \partial_i H = 0. \quad (1.6)$$

Such metrics represent gravitational radiation propagating in one fixed direction, and in fact have holonomy consisting of just the translation subgroup  $\mathbb{R}^{n-2}$  of  $E(n-2)$ .

A spacetime admitting a covariantly constant null vector field is also said to admit a Bargmann structure because dimensional reduction on a covariantly-constant null Killing vector field  $n^\mu$  gives rise to a non-relativistic Newton-Cartan spacetime with a degenerate co-metric [25, 26]. Dimensional reduction on a null Killing vector that is not necessarily covariantly constant has been studied by [27].

Brinkmann waves necessarily have vanishing Ricci scalar  $R^\mu_\mu = 0$ . They can be used to obtain time-independent [28] and time-dependent [15] extremal Kaluza-Klein multi black holes, and by electric-magnetic duality, multi Kaluza-Klein monopole metrics with Euclidean transverse space sections. However, these extremal Kaluza-Klein black holes or multi Kaluza-Klein monopoles move in a cosmological Friedman-Lemaitre-Robertson-Walker background with spatial sections having a flat metric  $g_{ij}$ . Interestingly the  $AdS_n$  analogue of pp-waves, while admitting a Killing spinor, and hence being supersymmetric in the appropriate dimensions, have the full  $SO(n-1, 1)$  as holonomy group.

If we are to obtain solutions with non-vanishing spatial curvature from Einstein metrics in one higher dimension with reduced holonomy and with non-vanishing cosmological constant, we need to look at a holonomy group which is larger than the that of the Brinkmann waves. Now, the maximal proper subgroup of the Lorentz group  $SO(n-1, 1)$  is the  $1 + \frac{1}{2}(n-2)(n-3)$  subgroup  $\text{Sim}(n-2)$ , which leaves invariant a null direction<sup>2</sup>.

Thus we need to find examples of Einstein metrics  $g_{\mu\nu}$  such that  $R_{\mu\nu} = \Lambda g_{\mu\nu}$  and with holonomy  $\text{Sim}(n-2)$ , and especially, those with non-vanishing cosmological constant  $\Lambda$ . This problem appears not to have been greatly studied.

In four dimensions,  $n = 4$ , and if the Einstein equations hold, holonomy  $\text{Sim}(2)$  implies that the Weyl tensor is of Petrov-Plebanski type *III*, whilst holonomy  $\mathbb{R}^2$  implies that it is of type *N* [29]. The latter can occur only if  $\Lambda = 0$ . Holonomy  $\text{Sim}(2)$  can occur both for  $\Lambda = 0$  and for  $\Lambda \neq 0$ ; an example of the latter is presented in [30]<sup>3</sup>. All solutions with  $\Lambda = 0$  and holonomy  $\text{Sim}(2)$  have been found, up to a solution of two linear equations [31, 32]. In the literature [29], the Lie algebras  $\mathfrak{sim}(2)$  and  $\mathbb{R}^2$ , thought of as sub-algebras of the Lorentz algebra  $\mathfrak{so}(3, 1)$ , are sometimes denoted by  $R_{14}$  and  $R_7$  respectively. In five dimensions, all vacuum metrics with  $E(3)$  holonomy, or a subgroup thereof, were written down by Brinkmann [24]. They depend on two functions that are harmonic in the three transverse variables but which are otherwise arbitrary.

If the functions in the five-dimensional Brinkmann metrics are taken to depend only on the transverse variables, one may dimensionally reduce to four dimensions and obtain stationary solutions of dilaton Einstein Maxwell gravity with dilaton-Maxwell coupling characterised by  $a = \sqrt{3}$  [28]. The case with for which  $A_i =$  was used in [28] to construct multi-electrically charged extreme black-hole solutions. The electric-magnetic dual then yielded the *Riemannian* multi-Taub-NUT solutions [4, 5] which may be interpreted as multi Kaluza-Klein monopoles [6, 7].

The case with  $A_i \neq 0$ , which requires two harmonic functions, was reduced to four dimensions to obtain multi- *Lorentzian* Taub-Nut solutions [26].

Another type of reduction was introduced in [21], in which the  $u$ -dependence is non-

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<sup>2</sup>Some more details about  $\text{Sim}(n-2)$  are given in the appendix.

<sup>3</sup>Note, however, that the metric presented in [30] has misprints; see later.

trivial but chosen so as to make the metric invariant under the the  $SO(1, 1)$  action  $u \rightarrow \lambda u$ ,  $v \rightarrow \lambda^{-1}v$ . Reduction on the boost Killing vector gives *time-dependent* multi-centre metrics of a type first constructed by Kastor and Traschen [13] and generalised by Maki and Shiraishi [14]. These have recently been used to discuss the collisions of branes [15, 16]. In particular, five-dimensional Brinkmann waves were used in [15] to construct time-dependent multi Kaluza-Klein monopole solutions.

In this paper we shall reduce the general problem of finding  $n$ -dimensional Einstein metrics with holonomy  $\text{Sim}(n - 2)$  to solving a linear system of Laplace-like and Poisson-like linear equations on an  $(n - 2)$ -dimensional transverse Einstein metric  $g_{ij}(x^k, u)$ ,  $R_{ij} = \Lambda g_{ij}$ , which can in general be time dependent. We give some explicit examples and discuss their dimensional reductions to give time-dependent extremal Kaluza-Klein black holes moving in a Friedman-Lemaitre-Robertson-Walker background with curved spatial sections and dominated by a scalar field with a Liouville potential.

It is striking that despite the historical sequence in which examples in four dimensions with  $\text{Sim}(2)$  holonomy were found, it is actually quite a lot simpler to obtain examples with  $\Lambda$  non-zero than it is to obtain  $\Lambda = 0$  examples.

The paper is organised as follows. In section 2 we describe the relationship between  $\text{Sim}(n - 2)$  holonomy and the existence of a recurrent null vector, leading us to a local form of the most general metric with this holonomy which was first written down by A G Walker. In section 3 we discuss the implications of  $\text{Sim}(n - 2)$  holonomy for the existence of special kinds of spinor fields. In section 4 we impose the Einstein equations on the Walker metrics and reduce them to solving a linear system of equations in the  $(n - 2)$ -dimensional transverse metric with its Einstein metric. In section 5, after describing the general reduction technique, we use the solutions we have obtained to construct time-dependent multi-centre metrics in 3+1 spacetime dimensions. In section 6 we discuss the circumstances under which the time dependence of the transverse metric may be eliminated. Our conclusions are contained in section 7. The appendix contains a short, self-contained and unified description of the Galilei, Bargmann, Carroll and Sim and ISim groups appearing in the paper as subgroups of a higher dimensional Poincare group.

## 2 Holonomy $\text{Sim}(n - 2)$ and recurrent null vector fields

A metric with holonomy  $\text{Sim}(n - 2)$  is by definition one which admits a null vector field  $n^\mu$ ,  $g_{\mu\nu}n^\mu n^\nu = 0$ , whose direction remains invariant under parallel transport. This means that the null vector  $n^\mu$  is ‘recurrent,’ i.e. it satisfies

$$\nabla_\mu n^\nu = B_\mu n^\nu, \quad (2.1)$$

for some ‘recurrence 1-form’  $B_\mu$ . Note that there is a gauge freedom, since  $n^\mu$  and  $\Omega n^\mu$  define the same null direction field. Under such a rescaling the recurrence form changes as

$$B \rightarrow B - d\Omega. \quad (2.2)$$

If we use the metric to convert  $n^\mu$  to a 1-form  $n_\mu dx^\mu = g_{\mu\nu} n^\nu dx^\mu$ , we may, by skew-symmetrising (2.1), deduce that

$$dn = B \wedge n, \quad (2.3)$$

and hence that

$$n \wedge dn = 0. \quad (2.4)$$

It follows from Frobenius’ theorem (see, for example, [33]) that

$$n = f du, \quad (2.5)$$

for functions  $f$  and  $u$ . Using the rescaling freedom we may set  $f = 1$  and thus

$$n = du, \quad dn = 0. \quad (2.6)$$

Now (2.3) implies that  $B = \kappa n$  for some function  $\kappa$ , and so (2.1) becomes

$$\nabla_\mu n_\nu = \kappa n_\mu n_\nu. \quad (2.7)$$

It follows that  $n^\mu$  is tangent to an affinely-parameterised, non-expanding, twist-free null geodesic congruence<sup>4</sup>. The null vector field  $n^\mu$  is tangent to the null generators of the null hypersurfaces  $u = \text{constant}$ .

We may now introduce as coordinates the function  $u$  and the affine parameter  $v$ , such that

$$n^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial v}, \quad (2.8)$$

together with  $n - 2$  transverse coordinates  $x^i$ ,  $i = 1, 2, \dots, n - 2$ . The metric may now be cast into the form introduced by Walker<sup>5</sup> [35]:

$$ds^2 = 2dudv + g_{ij}(u, x^k) dx^i dx^j + H(u, v, x^i) du^2 + 2A_i(u, x^j) dx^i du. \quad (2.9)$$

where  $g_{ij}(u, x^k)$ ,  $A_i(u, x^k)$ , and  $H(u, v, x^i)$  are arbitrary functions of their arguments. We shall shortly constrain these functions by imposing the Einstein vacuum field equations with cosmological term,  $R_{\mu\nu} = \Lambda g_{\mu\nu}$ .

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<sup>4</sup>This distinguishes metrics of  $\text{Sim}(n - 2)$  from the much less rich higher-dimensional Robinson-Trautman solutions, which admit an expanding twist-free shear-free null geodesic congruence [34].

<sup>5</sup>Not to be confused with the Friedman-Lemaitre-Robertson-Walker metrics.

Defining  $x^+ = v$  and  $x^- = u$ , implying  $n^\mu = \delta_+^\mu$  and  $n_\mu = \delta_\mu^-$ , and denoting partial derivatives by

$$f' \equiv \frac{\partial f}{\partial v}, \quad \dot{f} \equiv \frac{\partial f}{\partial u}, \quad \partial_i f \equiv \frac{\partial f}{\partial x^i}, \quad (2.10)$$

one finds that

$$\kappa = \frac{1}{2}H', \quad (2.11)$$

and so

$$\nabla_\mu n_\nu = \frac{1}{2}H' n_\mu n_\nu. \quad (2.12)$$

The Ricci identity

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) n_\tau = R_{\tau\sigma\mu\nu} n^\sigma \quad (2.13)$$

then implies that

$$R_{i+\mu\nu} = 0 \quad R_{-++-} = \frac{1}{2}H''. \quad (2.14)$$

The special case where  $H$  does not depend on the affine parameter  $v$ , and so  $\kappa = \frac{1}{2}H' = 0$ , corresponds to Brinkmann waves [22–24]. In this case the null vector field  $n^\mu$  is covariantly constant and hence  $\frac{\partial}{\partial v}$  becomes a null Killing vector field. The holonomy group is then reduced further, to  $E(n-2)$  or a subgroup thereof.

## 3 Holonomy and Spinors

### 3.1 Calculation of local holonomy

The local holonomy algebra is generated by the curvature tensor of the metric. The algebra is most easily seen if one uses a vielbein basis for the Riemann tensor. For the Walker metrics (2.9) a suitable basis, in which the metric takes the form  $ds^2 = 2\hat{e}^+\hat{e}^- + \hat{e}^\alpha\hat{e}^\alpha$ , is given by

$$\begin{aligned} \hat{e}^+ &= dv + A_i dx^i + \frac{1}{2}H du, \\ \hat{e}^- &= du, \\ \hat{e}^\alpha &= e^\alpha, \end{aligned} \quad (3.1)$$

where  $e^\alpha$  is a vielbein for the transverse metric,  $e^\alpha e^\alpha = g_{ij} dx^i dx^j$ . It is useful also to record that the inverse vielbein is given by

$$\begin{aligned} \hat{E}_+ &= \frac{\partial}{\partial v}, \\ \hat{E}_- &= \frac{\partial}{\partial u} - \frac{1}{2}H \frac{\partial}{\partial v}, \\ \hat{E}_\alpha &= E_\alpha - A_\alpha \frac{\partial}{\partial v}. \end{aligned} \quad (3.2)$$

We shall just present the expressions that give the non-vanishing vielbein components  $\hat{R}_{abcd}$  of Riemann tensor of (2.9) in the case where the functions  $H$ ,  $g_{ij}$  and  $A_i$  in (2.9) are taken to be independent of  $u$ . We then find

$$\begin{aligned}
\hat{R}_{+-+-} &= -\frac{1}{2}H'', \\
\hat{R}_{-\alpha+-} &= \frac{1}{2}(\nabla_\alpha H' - A_\alpha H''), \\
\hat{R}_{-\alpha-\beta} &= -\frac{1}{2}\nabla_\alpha \nabla_\beta H - \frac{1}{2}H'' A_\alpha A_\beta + \frac{1}{4}F_{\alpha\gamma}F_{\beta}{}^\gamma + \frac{1}{4}H'(\nabla_\alpha A_\beta + \nabla_\beta A_\alpha) \\
&\quad + \frac{1}{2}(A_\alpha \nabla_\beta H' + A_\beta \nabla_\alpha H'), \\
\hat{R}_{-\alpha\beta\gamma} &= \frac{1}{2}\nabla_\alpha F_{\beta\gamma}, \\
\hat{R}_{\alpha\beta\gamma\delta} &= R_{\alpha\beta\gamma\delta}.
\end{aligned} \tag{3.3}$$

Note, in particular, that all components of the form  $\hat{R}_{+abc}$  vanish, where  $b = (+, -, \beta)$  and  $c = (+, -, \gamma)$ . This means that the local holonomy is generated just by the  $\text{Sim}(n-2)$  subset of the Lorentz generators  $M_{ab}$ , comprising

$$M_{+\alpha}, \quad M_{+-}, \quad M_{\alpha\beta}. \tag{3.4}$$

(In other words, all generators except  $M_{-\alpha}$ .)

It is clear that in the more complicated case where  $H$ ,  $A_i$  and  $g_{ij}$  in (2.9) are allowed also to depend on  $u$  the set of non-vanishing Riemann tensor components will be at least as large as in the  $u$ -independent case (3.3) that we have presented explicitly. On the other hand, it is also clear that the components  $\hat{R}_{+abc}$  will continue to vanish when  $u$ -dependence is included. This follows from recurrence condition  $\nabla_\alpha n_\beta = \frac{1}{2}H' n_\alpha n_\beta$ , where the null vector  $n$  is given by  $n = du = e^-$ , by applying a second derivative and forming a commutator, as discussed previously. Therefore the local holonomy is again  $\text{Sim}(n-2)$  for the general Walker metrics (2.9) that include  $u$ -dependence.

### 3.2 Spinors in $\text{Sim}(n-2)$ metrics

In any of the metrics (2.9) of  $\text{Sim}(n-2)$  holonomy there exists a preferred spinor which generalises the covariantly-constant spinor that exists in the Brinkmann wave metrics. In the Brinkmann case, if the Einstein vacuum equations with  $\Lambda = 0$  hold within the framework of a supergravity theory, then this spinor generates half of the maximum number of supersymmetries. By contrast, although there still exists a preferred spinor in the more general  $\text{Sim}(n-2)$  holonomy metrics we are considering, this is not associated with any supersymmetry.

In the vielbein basis (3.1), a calculation of the spin connection shows that the Lorentz-covariant exterior derivative acting on spinors is given by

$$\hat{D} \equiv d + \frac{1}{4}\hat{\omega}_{ab}\Gamma^{ab}$$

$$\begin{aligned}
&= d + \frac{1}{4}\omega_{\alpha\beta}\Gamma^{\alpha\beta} + (\frac{1}{4}F_{\alpha\beta} - \frac{1}{4}\dot{g}_{ij}E_{\alpha}^iE_{\beta}^j)\Gamma^{-\alpha}\hat{e}^{\beta} \\
&+ \left[ (\frac{1}{4}\nabla_{\alpha}H - \frac{1}{4}H'A_{\alpha} - \frac{1}{2}\dot{A}_iE_{\alpha}^i)\Gamma^{-\alpha} - \frac{1}{4}H'\Gamma^{+-} + \frac{1}{8}(B_{\alpha\beta} - F_{\alpha\beta})\Gamma^{\alpha\beta} \right] \hat{e}^{-},
\end{aligned} \tag{3.5}$$

where

$$B_{\alpha\beta} \equiv \frac{1}{2}(\dot{e}_{i\beta}E_{\alpha}^i - \dot{e}_{i\alpha}E_{\beta}^i). \tag{3.6}$$

It is now evident that if we consider a spinor  $\epsilon$  that satisfies

$$\Gamma^{-}\epsilon = 0, \tag{3.7}$$

and that is independent of  $u$  and  $v$ , then the Dirac operator  $\Gamma^a\hat{D}_a$  acting on  $\epsilon$  reduces to the Dirac operator  $\Gamma^{\alpha}D_{\alpha}$  in the internal space:

$$\Gamma^a\hat{D}_a\epsilon = \Gamma^{\alpha}D_{\alpha}\epsilon, \tag{3.8}$$

where  $D \equiv d + \frac{1}{4}\omega_{\alpha\beta}\Gamma^{\alpha\beta}$ . In particular, if the internal space is maximally-symmetric, with

$$R_{\alpha\beta\gamma\delta} = \frac{\Lambda}{n-3}(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}), \tag{3.9}$$

then it admits a Killing spinor  $\epsilon$  satisfying

$$D_{\alpha}\epsilon = \sqrt{-\frac{\Lambda}{4(n-3)}}\Gamma_{\alpha}\epsilon. \tag{3.10}$$

This spinor therefore satisfies a massive Dirac equation in the full  $n$ -dimensional metric of  $\text{Sim}(n-2)$  holonomy, with

$$\Gamma^a\hat{D}_a\epsilon = (n-2)\sqrt{-\frac{\Lambda}{4(n-3)}}\epsilon. \tag{3.11}$$

The spinor  $\epsilon$ , which we take to be commuting, can be viewed as the square root of the distinguished null vector  $n^{\mu}$ , and with a suitable normalisation we have

$$n^{\mu} = \bar{\epsilon}\Gamma^{\mu}\epsilon. \tag{3.12}$$

## 4 Einstein Equations and New $\text{Sim}(n-2)$ Holonomy Metrics

### 4.1 The Einstein equations

After some algebra, we find that the non-vanishing coordinate-frame components  $\hat{R}_{\mu\nu}$  of the Ricci tensor of the general Walker class (2.9) of  $\text{Sim}(n-2)$  holonomy metrics are given

by

$$\begin{aligned}\hat{R}_{--} = & -\frac{1}{2}\left[\nabla^2 H + \partial_u(g^{ij}\dot{g}_{ij}) - \frac{1}{2}F^{ij}F_{ij} + \frac{1}{2}\dot{g}^{ij}\dot{g}_{ij} + g^{ij}\ddot{g}_{ij} - 2\nabla^i\dot{A}_i\right. \\ & \left.- 2A^i\partial_i H' - (\nabla^i A_i)H' - HH'' + \frac{1}{2}g^{ij}\dot{g}_{ij}H' + A^i A_i H''\right],\end{aligned}\quad (4.1)$$

$$\hat{R}_{-i} = \frac{1}{2}\partial_i H' + \frac{1}{2}\nabla^j F_{ij} + \frac{1}{2}\nabla^j \dot{g}_{ij} - \frac{1}{2}\partial_i(g^{jk}\dot{g}_{jk}), \quad (4.2)$$

$$\hat{R}_{+-} = \frac{1}{2}H'', \quad (4.3)$$

$$\hat{R}_{ij} = R_{ij}, \quad (4.4)$$

where we have defined

$$F_{ij} \equiv \partial_i A_j - \partial_j A_i. \quad (4.5)$$

(Note that  $\dot{g}^{ij}$  means  $\partial g^{ij}/\partial u$ , which is the same as  $-g^{ik}g^{j\ell}\dot{g}_{k\ell}$ .)

The Einstein equation  $\hat{R}_{+-} = \Lambda \hat{g}_{+-}$  implies

$$\frac{1}{2}H'' = \Lambda, \quad (4.6)$$

which has the immediate consequence that the  $v$ -dependence of  $H$  must be restricted to have the form

$$H(u, v, x^i) = \Lambda v^2 + v H_1(u, x^i) + H_0(u, x^i). \quad (4.7)$$

Thus the full system of Einstein equations  $\hat{R}_{\mu\nu} = \Lambda \hat{g}_{\mu\nu}$  implies that

$$\begin{aligned}\nabla^2 H_0 - \frac{1}{2}F^{ij}F_{ij} - 2A^i\partial_i H_1 - H_1\nabla^i A_i + 2\Lambda A^i A_i - 2\nabla^i\dot{A}_i \\ + \frac{1}{2}\dot{g}^{ij}\dot{g}_{ij} + g^{ij}\ddot{g}_{ij} + \frac{1}{2}g^{ij}\dot{g}_{ij}H_1 = 0,\end{aligned}\quad (4.8)$$

$$\nabla^j F_{ij} + \partial_i H_1 - 2\Lambda A_i + \nabla^j \dot{g}_{ij} - \partial_i(g^{jk}\dot{g}_{jk}) = 0, \quad (4.9)$$

$$\nabla^2 H_1 - 2\Lambda\nabla^i A_i + \Lambda g^{ij}\dot{g}_{ij} = 0, \quad (4.10)$$

$$R_{ij} = \Lambda g_{ij}. \quad (4.11)$$

The coordinate  $u$  plays the rôle of a modulus in the metrics  $g_{ij}(u, x^k)$ , in the sense that (4.11) must hold for all values of  $u$ . Note that (4.10) is redundant, since it follows by taking the divergence of (4.9). This can be seen from the fact that under an infinitesimal deformation of the metric, the Ricci tensor satisfies  $g^{ij}\delta R_{ij} = (g_{ij}\nabla^2 - \nabla_i\nabla_j)\delta g^{ij}$ , and hence  $g^{ij}\dot{R}_{ij} = (\nabla^i\nabla^j - g^{ij}\nabla^2)\dot{g}_{ij}$ . Taking the derivative of (4.11) with respect to  $u$ , it then follows that the divergence of (4.9) gives (4.10).

There is a gauge symmetry of the Walker metrics (2.9), which we shall discuss in the case where the Einstein equations have been imposed. The form of the Einstein metric is preserved under the transformations

$$\begin{aligned}v &\longrightarrow v - f, & A_i &\longrightarrow A_i + \partial_i f, \\ H_1 &\longrightarrow H_1 + 2\Lambda f, & H_0 &\longrightarrow H_0 + H_1 f + \Lambda f^2 + 2\dot{f},\end{aligned}\quad (4.12)$$

where  $f$  is an arbitrary function of  $u$  and  $x^i$ .

## 4.2 Reduction to a linear system for $\dot{g}_{ij} = 0$

If we suppose that the Einstein metric  $g_{ij}$  on the transverse space is independent of  $u$ , then the remaining equations (4.8)–(4.9) can be reduced to a linear system. To see this, it is convenient to use the gauge freedom (4.12) to impose the condition

$$H_1 = -\nabla \cdot \mathbf{A} \quad (4.13)$$

for all  $u$ . This leaves the residual gauge freedom

$$A_i \longrightarrow A_i - \partial_i w, \quad (4.14)$$

where  $w(u, \mathbf{x})$  satisfies the wave equation

$$\nabla^2 w + 2\Lambda w = 0. \quad (4.15)$$

Substituting (4.13) into (4.9) shows that  $A_i$  satisfies

$$\nabla^2 A_i + \Lambda A_i = 0, \quad (4.16)$$

and so (4.14) implies that

$$\begin{aligned} \nabla^2 A_i + \Lambda A_i &\longrightarrow \nabla^2 A_i + \Lambda A_i + (\nabla^2 + \Lambda)\partial_i w \\ &= \nabla^2 A_i + \Lambda A_i + \partial_i(\nabla^2 w + 2\Lambda w) \\ &= \nabla^2 A_i + \Lambda A_i. \end{aligned} \quad (4.17)$$

This means that the residual gauge transformation (4.14) could be used to set one component of  $A_i$  to zero.

Given any solution  $A_i$  of (4.16), the remaining equation (4.8) becomes a Poisson equation for  $H_0$ , with a known source. The solution therefore contains an arbitrary additive  $u$ -dependent harmonic function  $U$  in  $H_0$ . In summary, the general solution therefore depends on  $(n-2)$  arbitrary  $u$ -dependent solutions  $A_i$  of the modified vector Laplace equation (4.16) plus the one arbitrary  $u$ -dependent harmonic function  $U$ . The gauge freedom (4.14) reduces this to  $(n-3)$  solutions  $A_i$  plus the harmonic function  $U$ .

Explicit solutions of (4.16) are not very easy to obtain in general if  $\Lambda \neq 0$ . However, it is perhaps worth remarking that any Killing vector field in an Einstein space automatically satisfies (4.16).

## 4.3 $\Lambda \neq 0$ holonomy $\text{Sim}(n-2)$ solutions with $A_i = 0$

A dramatic simplification of the linear system occurs in the case that  $A_i = 0$ . This yields a very simple explicit construction of  $n$ -dimensional Einstein metrics with proper  $\text{Sim}(n-2)$

holonomy, provided that  $\Lambda \neq 0$ . Thus, we can consider the following restricted class of metrics within the Walker ansatz (2.9):

$$d\hat{s}^2 = 2dudv + g_{ij}(\mathbf{x})dx^i dx^j + [\Lambda v^2 + H_0(u, \mathbf{x})]du^2, \quad (4.18)$$

where  $g_{ij}$  is an  $(n-2)$ -dimensional Einstein metric satisfying  $R_{ij} = \Lambda g_{ij}$ , and  $H(u, \mathbf{x})$  is an arbitrary  $u$ -dependent harmonic function

$$\nabla^i \nabla_i H_0 = 0. \quad (4.19)$$

The metrics (4.18) are Einstein, satisfying  $R_{\mu\nu} = \Lambda g_{\mu\nu}$ , and they have proper  $\text{Sim}(n-2)$  holonomy provided that  $\Lambda \neq 0$ , that  $g_{ij}$  has maximal holonomy  $SO(n-2)$ , and that the harmonic function  $H_0$  is generic.<sup>6</sup>

We can see the nature of the local holonomy explicitly by looking at the curvature. Choosing the natural vielbein  $\hat{e}^a$  with

$$\hat{e}^+ = dv + \frac{1}{2}(\Lambda v^2 + H_0)du, \quad \hat{e}^- = dv, \quad \hat{e}^\alpha = e^\alpha, \quad (4.20)$$

we find that the non-zero components of the Riemann tensor are given by

$$\hat{R}_{+-+-} = -\Lambda, \quad \hat{R}_{-\alpha-\beta} = -\frac{1}{2}\nabla_\alpha \nabla_\beta H_0, \quad \hat{R}_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}. \quad (4.21)$$

In particular, all components of the form  $\hat{R}_{+\alpha bc}$  are zero, which proves that the local holonomy is either the full  $\text{Sim}(n-2)$  or a subgroup thereof. Proper holonomy  $\text{Sim}(n-2)$  arises provided that  $\hat{R}_{+-+-} \neq 0$  (and hence  $\Lambda \neq 0$ ), that  $g_{ij}$  has  $SO(n-2)$  holonomy, and that  $H_0$  is such that there are non-zero derivatives  $\nabla_\alpha \nabla_\beta H_0$ .

## 4.4 Previous results

### 4.4.1 Goldberg-Kerr $\Lambda = 0$ metrics in $n = 4$

Goldberg and Kerr constructed a class of 4-dimensional Ricci-flat metrics with proper  $\text{Sim}(2)$  holonomy [31, 32]. They showed that the Weyl tensor must be of Petrov type III, and that the metric could be cast in the form

$$ds^2 = 2dudv + dx^2 + dy^2 + 2\rho dxdu + (w - v\rho_x)du^2 \quad (4.22)$$

with  $\rho$  and  $w$  being functions of  $x$ ,  $y$  and  $u$ . This therefore corresponds to a specialisation of the discussion of Walker metrics that we gave in section 4.2, with  $n = 4$  and  $\Lambda = 0$ , and with

$$A_1 = \rho, \quad A_2 = 0, \quad H_0 = w. \quad (4.23)$$

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<sup>6</sup>If, for example,  $H_0$  were independent of the coordinates  $x_i$  then the metric  $d\hat{s}^2$  would simply be the direct sum of two-dimensional de Sitter or anti-de Sitter spacetime and the  $(n-2)$ -dimensional Einstein metric  $g_{ij}$ . This would have holonomy  $SO(1, 1) \times \text{Hol}(g_{ij})$ , which is a proper subgroup of  $\text{Sim}(n-2)$ .

(i.e. the gauge transformations have been used to set  $H_1 = -\nabla \cdot A$  and  $A_2 = 0$ .) Ricci flatness therefore implies

$$\rho_{xx} + \rho_{yy} = 0, \quad (4.24)$$

$$\omega_{xx} + \omega_{yy} = 2\rho_{-x} - 2\rho\rho_{xx} - (\rho_x)^2 + (\rho_y)^2. \quad (4.25)$$

The null vector  $n^\mu$  is equal to  $\delta_+^\mu$ , and from (2.12) we have

$$\nabla_\mu n_\nu = -\frac{1}{2}\rho_x n_\mu n_\nu. \quad (4.26)$$

We may introduce a Majorana representation for the four-dimensional Clifford algebra, by defining

$$\begin{aligned} \Gamma^0 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & \Gamma^1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ \Gamma^2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, & \Gamma^3 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (4.27)$$

If we then let

$$\epsilon = \begin{pmatrix} \alpha \\ \beta \\ \alpha \\ \beta \end{pmatrix} \quad (4.28)$$

for constants  $\alpha$  and  $\beta$ , we find that

$$\bar{\epsilon}\Gamma^\mu\epsilon = 2\sqrt{2}(\alpha^2 + \beta^2)n^\mu \quad (4.29)$$

and

$$\nabla_\mu \epsilon = -\frac{1}{4}n_\mu \begin{pmatrix} \alpha\rho_x + \beta\rho_y \\ \beta\rho_x - \alpha\rho_y \\ \alpha\rho_x + \beta\rho_y \\ \beta\rho_x - \alpha\rho_y \end{pmatrix}. \quad (4.30)$$

In other words, we find that

$$\nabla_\mu \epsilon = -\frac{1}{4}n_\mu (\rho_x - \rho_y \Gamma_{12})\epsilon. \quad (4.31)$$

If one recalls that the Clifford algebra  $\text{Clif}(3,1)$  is isomorphic with  $\text{Mat}_4(\mathbb{R})$  the algebra of  $4 \times 4$  real matrices which is spanned by the six antisymmetric matrices  $C, s\Gamma_5, C\Gamma_5\Gamma_\mu$  and

the ten symmetric matrices  $C\Gamma_\mu, C\Gamma_{\mu\nu}$ ,<sup>7</sup> one sees that the only bilinears one may construct from the commuting Majorana spinor are

$$n^\mu = \bar{\epsilon}\Gamma^\mu\epsilon, \quad (4.32)$$

$$F_{\mu\nu} = \bar{\epsilon}\Gamma_{\mu\nu}\epsilon. \quad (4.33)$$

One finds that the only non-vanishing components of  $F_{\mu\nu}$  are  $F_{-i}$ . Thus  $F_{\mu\nu}$  is null and simple  $F_{\mu\nu}F^{\mu\nu} = 0 = F_{\mu\nu} \star F^{\mu\nu} = 0$ , with

$$F_{\mu\nu}n^\mu = 0. \quad (4.34)$$

Since

$$\nabla_\rho F_{\mu\nu} = 2\alpha F_{\mu\nu}n_\rho + \beta n_\rho(g_{\mu 1}F_{\nu 2} + g_{\nu 2}F_{\mu 1} - g_{\mu(2)}F_{\nu 1} - g_{\nu 1}F_{\mu 2}), \quad (4.35)$$

it follows that  $F_{\mu\nu}$  is a ('test ') solution of Maxwell's equations

$$\nabla_{[\rho}F_{\mu\nu]} = 0, \quad (4.36)$$

$$\nabla^\rho F_{\rho\nu} = 0. \quad (4.37)$$

#### 4.4.2 Ghanam-Thompson $\Lambda \neq 0$ metric in $n = 4$

Only one  $\Lambda \neq 0$  solution with proper  $\text{Sim}(n-2)$  holonomy has appeared previously, namely a  $\text{Sim}(2)$  holonomy Einstein metric in  $n = [30]$ . This metric, after correcting typographical errors in [30], is

$$\begin{aligned} ds_4^2 = & 2dudv + \frac{dx^2 + dy^2}{2x^2} + 12xydxdu + 6(x^2 - y^2)dydu \\ & - [2v^2 + 4y(3x^2 - y^2)v + (x^2 + y^2)^3]du^2. \end{aligned} \quad (4.38)$$

In fact the potential  $A_i$  in this solution is pure gauge, and by making the coordinate transformation  $v \rightarrow v - y(3x^2 - y^2)$  one obtains the simpler metric form

$$ds_4^2 = 2dudv + \frac{dx^2 + dy^2}{2x^2} - [2v^2 + (x^2 - y^2)(x^4 - 14x^2y^2 + y^4)]du^2. \quad (4.39)$$

This four-dimensional example, as rewritten in the form (4.39), falls within the general class of  $n$ -dimensional Einstein metrics that we found in section 4.3, for a particular choice of  $H_0$  that is harmonic in the 2-dimensional hyperbolic Einstein metric  $(dx^2 + dy^2)/(2x^2)$ , with  $\Lambda = -2$ .

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<sup>7</sup>A basis may be chosen so that  $CT^0 = 1$ .

#### 4.4.3 Brinkmann waves in $n = 5$

The Goldberg-Kerr and Ghanam-Thompson metrics in  $n = 4$  dimensions have proper  $\text{Sim}(2)$  holonomy. No other Einstein metrics with proper  $\text{Sim}(n - 2)$  holonomy in any dimension  $n$  have previously been exhibited. One can of course also consider large classes of metrics whose holonomy is a proper subgroup of  $\text{Sim}(n - 2)$ . Our purpose here is not to give a comprehensive review of such metrics. However, there is one class that we do wish to mention, since we shall discuss them further in a later section. These metrics are the Brinkmann waves in  $n = 5$  dimensions, which have  $\mathbb{R}^3$  holonomy.

We take the transverse metric  $g_{ij}$  to be independent of  $u$ , and to be that of flat Euclidean space,  $g_{ij} = \delta_{ij}$ . Adopting standard the notation of 3-dimensional Cartesian vector analysis, with  $F_{ij} = \epsilon_{ijk}B_k$ , we have  $\mathbf{B} = \nabla \times \mathbf{A}$ . It follows from (4.4) that the metric is given by (2.9) with

$$\nabla \cdot \mathbf{A} = 0, \quad \nabla \times \mathbf{A} = \nabla V, \quad H = U + \frac{1}{2}V^2, \quad (4.40)$$

where  $U$  and  $V$  are arbitrary  $u$ -dependent harmonic functions in the transverse space, satisfying

$$\nabla^2 U = 0, \quad \nabla^2 V = 0. \quad (4.41)$$

## 5 Multi-Centre Metrics

In this section, we first show how the five-dimensional Brinkmann wave solutions of  $\mathbb{R}^3$  holonomy that we reviewed in section 4.4.3 can be used in order to construct stationary and also time-dependent multi-black-hole solutions upon dimensional reduction to four dimensions. We go on to generalise this construction by dimensionally reducing the  $n = 5$  specialisation of the  $\text{Sim}(n - 2)$  holonomy solutions with cosmological constant that we obtained in section 4.3.

### 5.1 Time-independent Kaluza-Klein black holes

If  $U$  and  $V$  are chosen to be independent of  $u$ , and if  $H > 0$ , then  $\partial/\partial u$  is a spacelike Killing vector field that can be used for performing a Kaluza-Klein reduction, using the standard formula

$$ds_5^2 = e^{-2\phi/\sqrt{3}} ds_4^2 + e^{4\phi/\sqrt{3}} (du + 2B)^2, \quad (5.1)$$

in which the lower-dimensional metric  $ds_4^2$  is in the Einstein conformal gauge. This leads to a stationary 4-dimensional metric in which the coordinate  $v$  plays the rôle of time. The four-dimensional metric is

$$ds_4^2 = -H^{-1/2}(dv + A)^2 + H^{1/2}dx^i dx^i, \quad (5.2)$$

and the Kaluza-Klein vector and scalar are given by

$$B = \frac{1}{2H} (dv + A), \quad e^{4\phi/\sqrt{3}} = H. \quad (5.3)$$

In the conventions we are using, the lower-dimensional Lagrangian is given by

$$\mathcal{L} = \sqrt{-g} \left( \frac{1}{4} R - \frac{1}{2} (\partial\phi)^2 - \frac{1}{4} e^{2\sqrt{3}\phi} G^2 \right), \quad (5.4)$$

where  $G = dB$ .

If we choose

$$V = 0, \quad H = 1 + \sum_{a=1}^N \frac{4M_a}{|\mathbf{x} - \mathbf{x}_a|}, \quad (5.5)$$

we obtain the metric describing  $N$  extremal Kaluza-Klein black holes with masses  $M_a$ , charges  $2M_a$ , and scalar charges  $\sqrt{3}M_a$  in equilibrium [28].

Adding a harmonic function  $V$  of the form

$$V = \sum_{a=1}^N \frac{N_a}{|\mathbf{x} - \mathbf{x}_a|} \quad (5.6)$$

would maintain the equilibrium by endowing these objects with NUT charges proportional to  $N_a$ . However, the resulting metrics would not be asymptotically flat. To obtain asymptotically flat metrics with angular momentum,  $V$  could be chosen to have the form of a sum of dipoles, the angular momenta being proportional to the dipole moments.

## 5.2 Time-dependent Kaluza-Klein black holes

To obtain time-dependent solutions in four dimensions, we choose  $U$  and  $V$  so that the five-dimensional metric is invariant under the  $SO(1,1)$  boost action  $u \rightarrow \lambda u$ ,  $v \rightarrow v/\lambda$ . To achieve this we introduce new coordinates  $t$  and  $z$  defined by

$$u = -\frac{2}{h} e^{-hz/2}, \quad v = t e^{hz/2}, \quad (5.7)$$

where  $h$  is an arbitrary constant, and we take  $U$ ,  $V$  and  $A_i$  to have the specific  $u$ -dependences

$$U(u, \mathbf{x}) = \frac{4\tilde{U}(\mathbf{x})}{h^2 u^2}, \quad V(u, \mathbf{x}) = \frac{2\tilde{V}(\mathbf{x})}{hu}, \quad A_i(u, \mathbf{x}) = -\frac{2\tilde{A}_i(\mathbf{x})}{hu}. \quad (5.8)$$

These  $u$ -dependences, which also imply that  $H(u, \mathbf{x}) = 4\tilde{H}(\mathbf{x})/(h^2 u^2)$  with  $\tilde{H} = \tilde{U} + \frac{1}{2}\tilde{V}^2$ , ensure that the five-dimensional metric is indeed boost invariant. It takes the form

$$ds_5^2 = -(\tilde{H} + ht)^{-1} (dt + \tilde{A})^2 + dx^i dx^i + (\tilde{H} + ht) \left( dz + \frac{dt + \tilde{A}}{\tilde{H} + ht} \right)^2, \quad (5.9)$$

and thus it reduces to give

$$\begin{aligned} ds_4^2 &= -(\tilde{H} + ht)^{-1/2}(dt + \tilde{A})^2 + (\tilde{H} + ht)^{1/2}dx^i dx^i, \\ B &= \frac{1}{2}(\tilde{H} + ht)^{-1}(dt + \tilde{A}), \quad e^{4\phi/\sqrt{3}} = \tilde{H} + ht \end{aligned} \quad (5.10)$$

in four dimensions.

These time-dependent solutions have a similar interpretation to the metrics introduced in [15]. One has extreme Kaluza-Klein black holes moving in a background  $k = 0$  FLRW universe dominated by a massless scalar field, and hence with scale factor  $a(\tau) \propto \tau^{1/3}$ .

### 5.3 Kaluza-Klein monopoles

The four-dimensional field equations admit electric-magnetic duality, whereby

$$\phi \longrightarrow -\phi, \quad G_{\mu\nu} \longrightarrow \overline{G}_{\mu\nu} = e^{2\sqrt{3}\phi} *G_{\mu\nu}, \quad g_{\mu\nu} \longrightarrow g_{\mu\nu}. \quad (5.11)$$

One finds that

$$\overline{G} = d\overline{B} = \frac{1}{2}(*_3\tilde{F}) \wedge (dt + \tilde{A}) - \frac{1}{2}*_3d\tilde{H} + \frac{1}{2}h*_3\tilde{A}. \quad (5.12)$$

Lifting back to five dimensions, this gives

$$ds_5^2 = -(dt + \tilde{A})^2 + (\tilde{H} + ht)dx^i dx^i + (\tilde{H} + ht)(dz + 2\overline{B})^2. \quad (5.13)$$

If  $h = 0$  and  $\tilde{V} = 0$ , we obtain [28] the static multi-monopole Kaluza-Klein 5-metric, which is the direct sum of a four-dimensional gravitational multi-instanton [4, 5] and time. This has holonomy  $SU(2)$  rather than the holonomy  $\mathbb{R}^3$  that we started with. The more general cases for which  $h$  and  $\tilde{V}$  are non-vanishing represent moving Kaluza-Klein monopoles which may have angular momenta and NUT charges.

### 5.4 Time-dependent cosmological black holes

If a spacelike Killing vector exists, then a dimensional reduction is still possible even if  $\Lambda \neq 0$ . The four-dimensional Lagrangian (5.4) then contains an additional Liouville potential for the scalar  $\phi$ :

$$\mathcal{L} = \sqrt{-g} \left( \frac{1}{4}R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{2\sqrt{3}\phi}G^2 - W(\phi) \right), \quad W(\phi) = \frac{3}{4}\Lambda e^{-2\phi/\sqrt{3}}. \quad (5.14)$$

We may take as our starting point the  $u$ -independent 5-dimensional metrics of the type we considered in section 4.3, i.e. where  $A_i = 0$  and  $H_0$  can be an arbitrary harmonic function on the transverse space. Since the metric  $g_{ij}$  is 3-dimensional and Einstein with  $\Lambda \neq 0$ , its universal cover must be either  $S^3$  or the hyperbolic space  $H^3$ , depending on whether  $\Lambda$  is

positive or negative. Using (5.1), the five-dimensional Einstein metric reduces to give the four-dimensional solution

$$\begin{aligned} ds_4^2 &= -\frac{dv^2}{(\Lambda v^2 + H_0)^{1/2}} + (\Lambda v^2 + H_0)^{1/2} g_{ij} dx^i dx^j, \\ \phi &= \frac{\sqrt{3}}{4} \log(\Lambda v^2 + H_0), \quad B = \frac{dv}{2(\Lambda v^2 + H_0)}. \end{aligned} \quad (5.15)$$

These metrics represent extremal charged black holes moving in a cosmological background.

Note that this reduction of  $u$ -independent five-dimensional solutions has yielded *time-dependent* metrics in four dimensions, in contrast to the reduction of  $u$ -independent Brinkmann solutions we performed in section (5.1).

It is convenient to define a rescaled transverse metric  $\hat{g}_{ij} = \frac{1}{2}|\Lambda|g_{ij}$ , so that the radius of curvature is  $k = \pm 1$ . If  $H_0$  is taken to be independent of  $x^i$  (i.e.  $H_0$  is a constant), then the metric (5.15) can be cast into the standard FLRW form by introducing a proper-time coordinate  $\tau$  according to

$$d\tau = \frac{dv}{(\Lambda v^2 + H_0)^{1/4}}, \quad (5.16)$$

and defining the scale factor

$$a(\tau) = \left( \frac{2}{|\Lambda|} \right)^{1/2} (\Lambda v^2 + H_0)^{1/4}. \quad (5.17)$$

The metric then be written as

$$ds_4^2 = -d\tau^2 + a^2(\tau) \hat{g}_{ij} dx^i dx^j. \quad (5.18)$$

The scale factor  $a(\tau)$  may be seen to satisfy the Friedman equation

$$\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{2}{3} \left( \frac{1}{2} \dot{\phi}^2 + W(\phi) \right), \quad (5.19)$$

and  $\phi$  satisfies

$$\frac{1}{a^3} \frac{d(a^3 \dot{\phi})}{d\tau} + \frac{dW(\phi)}{d\phi} = 0. \quad (5.20)$$

Note that here, we are using a dot to denote a derivative with respect to  $\tau$ . In fact the solution has the property that the derivative and the non-derivative terms in (5.19) are separately equal,

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{3} \dot{\phi}^2, \quad \frac{k}{a^2} = \frac{2}{3} W(\phi). \quad (5.21)$$

The equations of motion for a FLRW model of this kind can be derived from the Lagrangian

$$L = a^3 \left[ N^{-1} \left( \frac{1}{2} \dot{\phi}^2 - \frac{3}{2} \frac{\dot{a}^2}{a^2} \right) - N \left( W(\phi) + \frac{3k}{a^2} \right) \right], \quad (5.22)$$

where the lapse  $N$  is a Lagrange multiplier enforcing the constraint that the associated Hamiltonian  $\mathcal{H}$  vanishes

$$\mathcal{H} = a^3 \left[ \frac{1}{2} \dot{\phi}^2 - \frac{3}{2} \frac{\dot{a}^2}{a^2} + W(\phi) - \frac{3k}{a^2} \right] = 0. \quad (5.23)$$

The vanishing of the Hamiltonian  $\mathcal{H}$  is equivalent to the Friedman equation (5.19). The first order equations (5.21) resemble in some ways Bogomol'nyi equations but they appear not to be derivable from a super-potential.

As in our previous discussion of time-dependent solutions in section 5.2, instead of starting from  $u$ -independent 5-dimensional solutions and reducing on  $\partial/\partial u$  we may alternatively start from 5-dimensional solutions with the very specific  $u$ -dependence that ensures boost invariance under  $u \rightarrow \lambda u$ ,  $v \rightarrow v/\lambda$ . Again, the dimensional reduction is then performed by defining new coordinates as in (5.7), and then reducing on the spacelike Killing vector  $\partial/\partial z$ . Starting again with the 5-dimensional metrics considered in section 4.3, but now taking the harmonic function to have the form

$$H_0(u, \mathbf{x}) = \frac{4\tilde{H}_0(\mathbf{x})}{h^2 u^2}, \quad (5.24)$$

we obtain the metric

$$ds_4^2 = -\frac{dt^2}{(\Lambda t^2 + ht + \tilde{H}_0)^{1/2}} + (\Lambda t^2 + ht + \tilde{H}_0)^{1/2} g_{ij} dx^i dx^j \quad (5.25)$$

after dimensional reduction. This is in fact equivalent to the previous metric in (5.15), as can be seen by performing the redefinitions

$$t \longrightarrow v - \frac{h}{2\Lambda}, \quad \tilde{H}_0 \longrightarrow H_0 + \frac{h^2}{4\Lambda^2}. \quad (5.26)$$

In contrast to the situation for our dimensional reductions of Brinkmann waves, where the reductions on  $\partial/\partial u$  in section 5.1 gave time-independent four-dimensional solutions, whilst reductions on the boost Killing vector  $\partial/\partial z$  in section 5.2 gave time-dependent solutions, we see that when  $\Lambda$  is non-zero both the  $\partial/\partial u$  and  $\partial/\partial z$  reductions give time-dependent solutions in four dimensions, and in fact the two reduction schemes give equivalent such solutions.

Another difference between the Brinkmann reductions and the cosmological reductions is that the former give rise to multi-centre black holes in a  $k = 0$  FLRW background, whilst the latter give rise to multi-centre black holes in  $k = \pm 1$  FLRW backgrounds.

## 6 Solutions with $u$ -Dependent Transverse Metrics

So far, we have considered only situations where the transverse metric  $g_{ij}$  is independent of  $u$ . However, we could take for  $g_{ij}$  an arbitrary curve in the space of Einstein metrics. If we were to do so, our general reduction of the remaining equations to a linear system is still possible, as long as the extra terms involving  $\dot{g}_{ij}$  are included as sources.

The curve of Einstein metrics may or may not be among metrics that are related by diffeomorphisms. In the former case, it seems likely that by means of a coordinate transformation one may pass to the case where the transverse metric is independent of  $u$ . A situation in which this is definitely the case is when the metric  $g_{ij}$  is flat, with

$$g_{ij}(u, \mathbf{x}) = \gamma_{ij}(u), \quad A_i = 0, \quad H_1 = 0, \quad (6.1)$$

and so

$$ds^2 = 2dudv + \gamma_{ij}(u) dx^i dx^j + H_0(u, \mathbf{x}) du^2. \quad (6.2)$$

We pass to new coordinates  $\tilde{x}^m$ ,  $\tilde{u}$  and  $\tilde{v}$ , given by

$$x^i = P^i_m(u) \tilde{x}^m, \quad v = \tilde{v} - \tilde{x}^m A_{mn}(u) \tilde{x}^n, \quad u = \tilde{u}, \quad (6.3)$$

where  $P^i_m$  is chosen such that

$$\gamma_{ij}(u) P^i_m(u) P^j_n(u) = \delta_{mn}. \quad (6.4)$$

Furthermore,  $A_{mn}(u)$ , which is symmetric, will be chosen to eliminate the resulting  $d\tilde{x}^m d\tilde{u}$  cross terms in the metric.

In what follows it will prove convenient to adopt a matrix notation and write this as

$$P^t \gamma P = 1, \quad (6.5)$$

where  $^t$  denotes transpose. This can equivalently be written as

$$\gamma = (PP^t)^{-1}, \quad (6.6)$$

from which it is clear that the choice of  $P$  is arbitrary up to an  $O(n-2)$  transformation  $U$ :

$$P' = PU. \quad (6.7)$$

We also define a matrix

$$B = P^{-1} \dot{P}, \quad (6.8)$$

where in this section  $\dot{P}$  denotes differentiation with respect to  $u$ .

The  $d\tilde{x}^m d\tilde{u}$  cross terms coming from the metric (6.2) will be absent if  $A$  can be chosen so that  $2A = P^t \gamma \dot{P}$ , which, using (6.5) and then (6.8), means

$$2A = P^{-1} \dot{P} = B. \quad (6.9)$$

In general,  $B = P^{-1} \dot{P}$  is not symmetric, unlike  $A$ . However, we can use the freedom to perform the  $u$ -dependent  $O(n-2)$  transformation (6.7) in order to find a suitable  $P'$  for which  $B' \equiv P'^{-1} \dot{P}'$  is symmetric. Specifically,  $U$  should be chosen so that

$$B^t - B = 2\dot{U}U^{-1}. \quad (6.10)$$

This always admits a solution for  $U$ , since there are  $\frac{1}{2}(n-2)(n-3)$  first-order equations for  $\frac{1}{2}(n-2)(n-3)$  unknown functions. Having seen that a gauge can be achieved where  $B'$  is symmetric, we may now drop the prime and assume that such a symmetric  $B$  exists.

Having eliminated the  $d\tilde{x}^m d\tilde{u}$  cross terms in the metric by choosing  $A$  to satisfy (6.9), the metric (6.2) in the tilded coordinates takes the form

$$ds^2 = 2d\tilde{u}d\tilde{v} + d\tilde{x}^m d\tilde{x}^m + \tilde{H} d\tilde{u}^2. \quad (6.11)$$

where

$$\tilde{H} = H_0 + \tilde{x}^m \tilde{K}_{mn} \tilde{x}^n, \quad (6.12)$$

and

$$\tilde{K} = \dot{P}^t \gamma \dot{P} - 2\dot{A}. \quad (6.13)$$

Using (6.5) and (6.9), we have  $\dot{P}^t \gamma \dot{P} = B^t P^t \gamma P B = B^2$ , and hence

$$\tilde{K} = B^2 - \dot{B}. \quad (6.14)$$

From (4.8), the function  $H_0$  satisfies

$$\gamma^{ij} \partial_i \partial_j H_0 + \frac{1}{2} \dot{\gamma}^{ij} \dot{\gamma}_{ij} + \gamma^{ij} \ddot{\gamma}_{ij} = 0 \quad (6.15)$$

in the original coordinate system. Using (6.6), (6.8) and (6.9), it can be seen that

$$\dot{\gamma}^{-1} \dot{\gamma} = -4PB^2P^{-1}, \quad \gamma^{-1} \ddot{\gamma} = 2P(2B^2 - \dot{B})P^{-1}, \quad (6.16)$$

and so (6.15) becomes

$$0 = \gamma^{ij} \partial_i \partial_j H_0 + 2\text{tr}(B^2 - \dot{B}) = \gamma^{ij} \partial_i \partial_j H_0 + 2\text{tr}\tilde{K}. \quad (6.17)$$

Transforming to the tilded coordinates (6.3), we see that the function  $\tilde{H}$  in the transformed metric (6.11) must simply obey the harmonic equation

$$\tilde{\partial}_m \tilde{\partial}_m \tilde{H} = 0. \quad (6.18)$$

## 6.1 The reverse transformation: Rosen waves

We may obviously carry out the previous steps in reverse. Suppose that

$$ds^2 = 2d\tilde{u}d\tilde{v} + \tilde{H}(\tilde{u}, \tilde{x}^i)d\tilde{u}^2 + d\tilde{x}^i d\tilde{x}^i. \quad (6.19)$$

Let

$$\tilde{x} = M(u)x \quad \tilde{v} = v + x^t N(u)x, \quad \tilde{u} = u. \quad (6.20)$$

where without loss of generality  $N(u)$  can be assumed to be symmetric. The transformed metric is

$$ds^2 = 2dudv + \gamma_{ij}(u)dx^i dx^j + H(u, x^i)du^2, \quad (6.21)$$

where

$$\gamma_{ij}(u) = M^t M, \quad (6.22)$$

$$H = \tilde{H} + x^t \left( 2\dot{N} + \dot{M}^t \dot{M} \right) x, \quad (6.23)$$

and the cross terms between  $dx^i$  and  $du$  are eliminated if  $N$  and  $M$  are such that

$$2N + M^t \dot{M} = 0. \quad (6.24)$$

Thus to remove the cross terms, it must be that  $M^t \dot{M}$  is symmetric. However, in the notation of the previous section,

$$M = P^{-1}, \quad N = M^t A M, \quad (6.25)$$

and

$$M^t \dot{M} = -M^t B M. \quad (6.26)$$

Thus we may use the same gauge freedom as before to arrange that  $M^t \dot{M}$  is symmetric.

Differentiation of (6.24) and elimination of  $\dot{N}$  from (6.23) gives

$$H = \tilde{H} - x^t (M^t \ddot{M}) x. \quad (6.27)$$

Now suppose that

$$\tilde{H} = \tilde{x}^t \tilde{K} \tilde{x} = x^t M^t K M x, \quad (6.28)$$

with  $K = L^t, \tilde{K} = \tilde{K}^t$ . We can then set  $H = 0$  by solving

$$\ddot{M} = K M. \quad (6.29)$$

Given  $L$ , this is a linear second order differential equation for  $M$  which can always be solved. The result is that in the  $x, u, v$  coordinates, the metric is independent of  $x$ , and the metric  $\gamma_{ij}$  satisfies the single condition vacuum field equation

$$\frac{1}{2}\dot{\gamma}^{ij}\dot{\gamma}_{ij} + \gamma^{ij}\ddot{\gamma}_{ij} = 0, . \quad (6.30)$$

Such solutions are called plane gravitational waves

The  $1 + 2(n - 2)$  vector fields

$$V = \partial_v, \quad P_i = \partial_i \quad (6.31)$$

and

$$U_i = x_i \partial_v + u \partial_i \quad (6.32)$$

generate isometries [11, 36]. Since, in general,  $\gamma_{ij}$  will have no isometries and so we omit

$$L_{ij} = x_i \partial_j - x_j \partial_i. \quad (6.33)$$

As a result the symmetry of the Rosen waves is the subgroup of the Carroll group [37, 38] (itself generated by  $U_i, P_i, V, L_{ij}$ ) which omits the rotations  $L_{ij}$ .

The reason for the name Carroll group is explained in the appendix.

## 6.2 Time dependence of curved transverse metrics

If

$$ds^2 = 2du \left( dv + A_i(u, x) dx^i \right) + g_{ij}(u, x) dx^i dx^j + H(u, v, x) du^2, \quad (6.34)$$

then under the coordinate transformation

$$v = \tilde{v} + F(u, \tilde{x}), \quad x^i = a^i(u, \tau, x), \quad (6.35)$$

we shall have

$$H \rightarrow H + 2\dot{F} + g_{ij}\dot{a}^i\dot{a}^j + 2A_i\dot{a}^i, \quad (6.36)$$

$$A_i \rightarrow A_r \frac{\partial x^r}{\partial \tilde{x}^i} + \frac{\pi F}{\pi \tilde{x}^i} + g_{rs}\dot{a}^r \frac{\pi x^s}{\pi \tilde{x}^i}, \quad (6.37)$$

$$g_{ij} \rightarrow g_{rs} \frac{\pi a^r}{\pi \tilde{x}^i} \frac{\pi a^s}{\pi \tilde{x}^j}. \quad (6.38)$$

Thus if  $g_{ij}(u, x)$  is, for each fixed  $u$ , a coordinate transformation of a fixed  $u$ -independent metric, then we can choose coordinates so that  $g_{ij}$  is independent of  $u$ , by choosing an appropriate  $a^i(u, \tilde{x})$ . If there is only one Einstein metric, with fixed  $\Lambda$ , up to diffeomorphisms on the base manifold, then this is always the case.

If, however, the space of Einstein metrics on the base manifold has non-trivial moduli,  $g_{ij}(u, x)$  could pass along a path of non-diffeomorphic metrics and the time-dependence cannot be eliminated in this way.

Indeed either the curve of metrics can be chosen arbitrarily, and for each  $u$  we can solve for  $A_i$ ,  $H_1$  and  $H_0$ , or we can set, for example,  $A_i = 0 = H_1 = H_0$  and try to find conditions on  $g_{ij}(u.x)$ . These might be consistent time-dependent Calabi-Yau's for example.

A separate question, discussed earlier in the case of flat metrics, is whether the cross-term  $A_i$  can be eliminated or kept non-zero by a suitable choice of  $F(u, \tilde{x})$ .

## 7 Conclusion

In this paper we have shown that the problem of finding the general  $n$ -dimensional Lorentzian Einstein metric with  $\text{Sim}(n-2)$  may be reduced to solving a set of linear equations on an  $(n-2)$ -dimensional transverse metric which itself is a, possibly time dependent, Einstein metric. We have also shown how the metric may be used to construct new four-dimensional multi-centre metrics in which extreme Kaluza-Klein black holes move in a background F-L-R-W metric with non-vanishing space curvature coupled to a scalar field with a Liouville potential.

We believe that the general  $\text{Sim}(n-2)$  holonomy metrics we have constructed in this paper will find various other applications in M-theory and string theory in the future.

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## A $\text{Sim}(n-2)$ , $\text{ISim}(n-2)$ and the Carroll Group

In this appendix we recall some facts about the Carroll group explained in [26] and relate them to the groups  $\text{Sim}(n-2)$ ,  $\text{ISim}(n-2)$ .

The Carroll group, and the Galilei group are both kinematic groups of a spacetime in the sense of [39, 40] and both may be regarded as the symmetry group of a structure in a Lorentzian spacetime with one higher dimensions.

We start with a construction of the Galilei group. The basic idea is to start with flat Minkowski spacetime  $\mathbb{E}^{n,1}$  whose metric written in double null coordinates  $(u, v, x^i)$ ,  $i = 1, 2, \dots, n-1$ , is

$$ds^2 = -2dudv + dx^i dx^i. \quad (\text{A.1})$$

The Lie algebra of the Poincaré group  $\mathfrak{e}(n, 1)$  is spanned by the Killing vector fields generating the Lie algebra of the Euclidean group  $\mathfrak{e}(n-1)$ , translations and rotations

$$P_i = \partial_i \quad L_{ij} = x_i \partial_j - x_j \partial_i, \quad (\text{A.2})$$

two null translations and one boost

$$U = \partial_u, \quad V = \partial_v \quad N = u\partial_u - v\partial_v, \quad (\text{A.3})$$

and two further sets of boosts

$$U_i = u\partial_i + x_i\partial_v \quad V_i = v\partial_i + x_i\partial_u. \quad (\text{A.4})$$

There is an obvious symmetry under inter-changing  $u$  and  $v$  induced by reflection in the timelike  $(n-1)$ - plane  $u = v$ .

To obtain the Bargmann group, the central extension of the Galilei group, we ask for the subgroup which commutes with the null translation generated by  $V = \partial_v$ . This is generated by  $\{P_i, L_{ij}, U, V, U_i, \}$ . The Galilei group is obtained by taking the quotient by the null translation group  $\mathbb{R}$  generated by  $V$ . It is easy to see that the Galilei group acts on the quotient  $\mathbb{E}^{n,1}/\mathbb{R}$  or light-like shadow, which may be identified with a Newton-Cartan spacetime  $M^n$ , the coordinate  $u$  playing the role of Newtonian absolute time. The generators  $U_i$  are Galilean boosts. Because

$$[P_i, U_j] = \delta_{ij}V, \quad (\text{A.5})$$

they commute with spatial translations (modulo  $V$ ) but not with time translations

$$[U, U_i] = P_i. \quad (\text{A.6})$$

One may regard this construction in terms of a Kaluza-Klein type reduction in which one think of  $\mathbb{E}^{n,1}$  as a fibre bundle with projection map

$$\pi : \mathbb{E}^{n,1} \rightarrow M^n \quad (\text{A.7})$$

given by  $(u, v, x^i) \rightarrow (u, x^i)$ . However in contrast to the usual case, the fibres are lightlike. Using the map  $\pi$  one may push forward the Minkowski co-metric on  $\mathbb{E}^{n-1}$  down to the Newton–Cartan  $\mathbb{E}_{n-1,0}$  spacetime to give the degenerate co-metric. More about lightlike reduction may be found in [27, 41]. For an interesting application of the inverse process, lightlike oxidation, see [42].

To obtain the Carroll group, we ask instead for the subgroup of the Poincaré group which leaves invariant the null hyperplane  $u = \text{constant} = 0$ . This is generated by  $\{P_i, L_{ij}, V, U_i, N\}$ . To obtain the Carroll group we quotient by the boosts  $N$ . Now the null coordinate  $v$  plays the role of time. The Carrollian boosts  $U_i$  commute with time translation

$$[V, U_i] = 0, \quad (\text{A.8})$$

but by (A.5) they no longer commute with spatial translations  $P_i$ . In fact one obtains a Heisenberg sub-algebra with the time translations being central. Note that the boosts  $N$ , which generate

$$v \rightarrow \lambda v, \quad u \rightarrow \lambda^{-1} u \quad (\text{A.9})$$

act as time dilations

$$t \rightarrow \lambda t, \quad x_i \rightarrow x_i. \quad (\text{A.10})$$

From an algebraic point of view the Carroll and Galilei groups differ only in the choice of generator of time translations: one picks either  $V$  or  $U$ .

One may think of the null hyperplane  $u = \text{constant}$  as the image under the embedding map

$$x : M^n \rightarrow \mathbb{E}^{n,1}, \quad (\text{A.11})$$

such that  $(v, x^i) \rightarrow (\text{constant}, v, x^i)$ , of a Carrollian spacetime time. The pull back of the Minkowski metric gives the degenerate Carrollian metric. Thus the duality relating the cases is between an immersion  $x$  (A.11) and a submersion  $\pi$  (A.7) and interchanges domain and range.

If we retain the generators  $U_i, N, L_{ij}$  we obtain an  $\frac{1}{2}(n^2 - 3n + 4)$ - dimensional subgroup of the Lorentz group  $SO(n - 1, 1)$  invariant and which normalises the lightlike vector field  $V$ ,

$$[N, V] = V. \quad (\text{A.12})$$

Since

$$[U_i, U_j] = 0, \quad [N, L_{ij}] = 0, \quad [N, U_i] = U_i, \quad (\text{A.13})$$

This group is isomorphic to the Euclidean group  $E(n - 2)$  augmented with homotheties, and is thus called  $\text{Sim}(n - 2)$ . Together with the translations  $p_i, U, V$  one obtains an  $\frac{1}{2}(n^2 - n + 4)$ - dimensional subgroup of the Poincaré group called  $\text{ISim}(n - 2)$  which, in the case  $n = 4$  is the basis of Very Special Relativity [43, 44] which is used as model of broken Lorentz-invariance with no invariant tensor fields, called in this context spurion fields.